Olympiades entraînement.

/47
+3 présentation orthographe

Dérivation dans un anneau.

I_1

Soient $(a,b) \in A^2$.

$$\langle a,b\rangle + \langle b,a\rangle = a\otimes b - b\otimes a + b\otimes a - a\otimes b$$
$$= a\otimes b - a\otimes b + b\otimes a - b\otimes a$$

$$\boxed{\langle a,b\rangle + \langle b,a\rangle = 0_A.}$$

(Cette dernière égalité équivaut successivement à :

$$\langle a,b\rangle + \langle b,a\rangle - \langle b,a\rangle = 0 - \langle b,a\rangle$$

$$\boxed{\langle a,b\rangle = -\langle b,a\rangle}$$

I_2

Soit $(a,b,c) \in A^3$.

$$\langle a, b+c\rangle = a\otimes(b+c) - (b+c)\otimes a.$$
$$= a\otimes b + a\otimes c - b\otimes a - c\otimes a$$
$$= a\otimes b - b\otimes a + a\otimes c - c\otimes a$$

$$\boxed{\langle a, b+c\rangle = \langle a,b\rangle + \langle a,c\rangle}$$

I_3

Soit $(a,b,c) \in A^3$.

$*$ $\langle a, \langle b,c\rangle\rangle$

$= a\otimes\langle b,c\rangle - \langle b,c\rangle\otimes a$

$= a\otimes(b\otimes c - c\otimes b) - (b\otimes c - c\otimes b)\otimes a$

$= a\otimes(b\otimes c) - a\otimes(c\otimes b) - (b\otimes c)\otimes a + (c\otimes b)\otimes a$

$= a\otimes b\otimes c - a\otimes c\otimes b - b\otimes c\otimes a + c\otimes b\otimes a$

* En permutant les lettres :

$<a,<b,c>> + <b,<c,a>> + <c,<a,b>>$

$= a\otimes b\otimes c - a\otimes c\otimes b - b\otimes c\otimes a + c\otimes b\otimes a$

$+ b\otimes c\otimes a - b\otimes a\otimes c - c\otimes a\otimes b + a\otimes c\otimes b$

$+ c\otimes a\otimes b - c\otimes b\otimes a - a\otimes b\otimes c + b\otimes a\otimes c$

On remarque un télescopage et :

$$<a,<b,c>> + <b,<c,a>> + <c,<a,b>> = 0_A.$$

I-4

Soient $(x,y)\in A^2$.

$d_a(x+y) = <a,x+y>$

$\quad = <a,x> + <a,y>$

$$d_a(x+y) = d_a(x) + d_a(y)$$

D'une part :

$d_a(x\otimes y) = <a, x\otimes y>$

$\quad = a\otimes x\otimes y - x\otimes y\otimes a$

d'autre part :

$x\otimes d_a(y) + d_a(x)\otimes y$

$= x\otimes <a,y> + <a,x>\otimes y$

$= x\otimes(a\otimes y - y\otimes a) + (a\otimes x - x\otimes a)\otimes y$

$= x\otimes a\otimes y - x\otimes y\otimes a + a\otimes x\otimes y - x\otimes a\otimes y$

$= a\otimes x\otimes y - x\otimes y\otimes a$

donc : $d_a(x\otimes y) = x\otimes d_a(y) + d_a(x)\otimes y$.

II-1

D'après (i) :

$D(0_A + 0_A) = D(0_A) + D(0_A)$

Or $0_A + 0_A = 0_A$ donc

$D(0_A) = D(0_A) + D(0_A)$

$$D(0_A) - D(0_A) = D(0_A) + D(0_A) - D(0_A)$$

$$\boxed{0_A = D(0_A)}$$

II - 2

De même, d'après (ii)

$$D(1_A \otimes 1_A) = 1_A \otimes D(1_A) + D(1_A) \otimes 1_A$$

$$D(1_A) = D(1_A) + D(1_A)$$

$$D(1_A) - D(1_A) = D(1_A) + D(1_A) - D(1_A)$$

$$\boxed{0 = D(1_A)}$$

II - 3

Soit $x \in A$.

$$D(0_A) = 0_A$$

$$D(-x + x) = 0_A$$

$$D(-x) + D(x) = 0_A$$

$$\boxed{D(-x) = -D(x)}$$

II - 4.

Soit $x \in A$, x inversible.

$$D(x \otimes x^{-1}) = x \otimes D(x^{-1}) + D(x) \otimes x^{-1}$$

$$D(1_A) = x \otimes D(x^{-1}) + D(x) \otimes x^{-1}$$

$$0_A = x \otimes D(x^{-1}) + D(x) \otimes x^{-1}$$

$$\cancel{D(x) \otimes x^{-1} = -x \otimes D(x^{-1})}$$

$$x^{-1} \otimes 0_A = x^{-1} \otimes \left(x \otimes D(x^{-1}) + D(x) \otimes x^{-1} \right)$$

$$0_A = x^{-1} \otimes x \otimes D(x^{-1}) + x^{-1} \otimes D(x) \otimes x^{-1}$$

$$0_A = D(x^{-1}) + x^{-1} \otimes D(x) \otimes x^{-1}$$

$$\boxed{-x^{-1} \otimes D(x) \otimes x^{-1} = D(x^{-1})}$$

II_5_a_

$*\ D(x^2) = D(x \otimes x)$

$$\boxed{D(x^2) = x \otimes D(x) + D(x) \otimes x}$$

$*\ D(x^3) = D(x \otimes x^2)$

$= \cancel{D(x)} \otimes D(x^2) + D(x) \otimes x^2$

$= x \otimes (x \otimes D(x) + D(x) \otimes x) + D(x) \otimes x^2$

$= x \otimes x \otimes D(x) + x \otimes D(x) \otimes x + D(x) \otimes x^2$

$$\boxed{D(x^3) = x^2 \otimes D(x) + x \otimes D(x) \otimes x + D(x) \otimes x^2}$$

II_5_b_i

$*\ \boxed{D(x^2) = 2x \otimes D(x)}$

$*\ \boxed{D(x^3) = 3\cancel{x}^2 D(x)}$

II_5_b_ii

On conjecture :

$$\boxed{\forall n \in \mathbb{N}^*, D(x^n) = n x^{n-1} D(x).}$$